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Nonlocal Models of Cosmic Acceleration

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Abstract I review a class of nonlocally modified gravity models which were proposed to explain the current phase of cosmic acceleration without dark energy. Among the topics considered are deriving causal and conserved field equations, adjusting the model to make it support a given expansion history, why these models do not require an elaborate screening mechanism to evade solar system tests, degrees of freedom and kinetic stability, and the negative verdict of structure formation. Although these simple models are not consistent with data on the growth of cosmic structures many of their features are likely to carry over to more complicated models which are in better agreement with the data.

Keywords Nonlocal · Dark Energy · Modified Gravity

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1 Introduction

In 1998 Hubble plots of distant Type Ia supernovae [1] revealed that the universe began accelerating somewhat over six billion years ago. Supernova data since then have continued to support this conclusion [2] and it has been confirmed by other data sets [3]. This triumph of observational cosmology continues to challenge fundamental theorists [4].

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The data are consistent with general relativity operating on a critical energy density whose current composition is about 70% cosmological constant, with about 30% nonrelativistic and small amounts of relativistic matter [5, 6]. However, fundamental theorists have a hard time understanding why the cosmological constant should be much smaller than any other scale in physics and why it should have the right magnitude to so recently come into dominance [7]. Scalar potential models [8, 9] can be devised to reproduce the observed expansion history [10, 11] but they are difficult to motivate. Quantum effects from a very light scalar have also been suggested [12].

Modifications of gravity have been considered [13]. However, generalizing the Hilbert Lagrangian from R to $f(R)$ [14] represents the only local, metric-based, generally coordinate invariant and stable modification of gravity [15]. Because the only model within this class which exactly reproduces the Λ CDM expansion history is general relativity with $f(R) = R - 2\Lambda$ [16], more general $f(R)$ models can deviate from observation even without considering perturbations.

There is greater freedom to modify gravity if locality is abandoned [17, 18, 19], but one should keep in mind Newton's famous dictum of 1692/3 [20]:

that one body may act upon another at a distance thro' a Vacuum, without the Mediation of any thing else, by and through which their Action and Force may be conveyed from one to another, is to me so great an Absurdity that I believe no Man who has in philosophical Matters a competent Faculty of thinking can ever fall into it.

The great man's misgivings have certainly stood the test of time: more than three centuries of observation and experimentation have failed to reveal a single exception to the paradigm of local, second order field equations. Proposed theoretical exceptions, such as string field theory, tend to have extra degrees of freedom which carry negative energy [21].

My view is that nonlocal modifications of gravity cannot be fundamental. If these models describe nature, I believe their nonlocal structures must derive from the gravitational vacuum polarization of infrared gravitons which were produced during the epoch of primordial inflation [22]. This is a natural expectation because accelerated expansion creates a vast ensemble of infrared gravitons — it causes the tensor power spectrum [23] — and because the large vacuum energy of primordial inflation provides a dimension three self-interaction [24, 25]. Of course loops of massless particles can produce long range effects, as witness confinement in quantum chromodynamics (QCD). Gravitons with zero vacuum energy have no impact on cosmology merely because the self-interaction has dimension five [26]. However, a dimension three coupling between massless particles ought to engender stronger infrared effects than the dimension four couplings of QCD [27].

Quantum gravitational self-interactions during inflation are suppressed by the small ($\sim 10^{-10}$) loop-counting parameter of quantum gravity, but they also grow with the number of inflationary e-foldings [28]. The growth occurs because more and more infrared gravitons come into causal contact the longer inflation persists. One consequence of it is that the interactions become nonperturbatively strong during a sufficiently prolonged period of inflation. That is frustrating because it means some sort of resummation

technique must be devised in order to gain quantitative control over late time cosmology. This task is not hopeless — it can be done for scalar models [29, 30] — but it has not been accomplished yet for quantum gravity [31]. In the absence of such a nonperturbative resummation technique the approach I shall describe here is the purely phenomenological one of proposing a plausible set of nonlocal field equations and examining their properties. However, it is well to note that models which might eventually be derived from inflationary vacuum polarization must possess two features which would be forbidden in fundamental theory:

- There is an initial value surface corresponding to inflation’s end; and
- Because the putative effect derives from cosmological-scale gravitons it is completely natural that there should no change in gravity on small scales.

The model I will describe in this article was developed with Stanley Deser. We change the gravitational Lagrangian from that of general relativity $\mathcal{L} = R\sqrt{-g}/16\pi G$ by the addition of a nonlocal term of the form [32],

$$\Delta\mathcal{L} \equiv \frac{1}{16\pi G} R\sqrt{-g} \times f\left(\frac{1}{\square}R\right). \quad (1)$$

Here $\square \equiv (-g)^{-\frac{1}{2}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu]$ is the scalar d’Alembertian and we define its inverse with retarded boundary conditions, which make $\frac{1}{\square}R$ and its first time derivative vanish at the initial time [32]. (Note the appearance of an initial time!) In addition to simplicity, the great advantage of this class of models is to provide a natural delay for the onset of cosmic acceleration: because the Ricci scalar R vanishes during radiation domination, $\frac{1}{\square}R$ cannot begin to grow until after the onset of matter domination, and then its growth is only logarithmic because of the inverse differential operator.

The model is defined by its field equations which take the form, $G_{\mu\nu} + \Delta G_{\mu\nu} = 8\pi GT_{\mu\nu}$. Section 2 explains how to derive conserved field equations that are nevertheless causal. Section 3 describes how to choose the function $f(X)$ in expression (1) to make the model reproduce the Λ CDM expansion history with zero cosmological constant. This construction only determines the shape of $f(X)$ for $X < 0$, leaving its behavior for $X > 0$ completely free. Section 4 exploits this freedom to avoid any deviation from general relativity inside gravitationally bound systems. (Note the absence of small scale modifications, as foretold above.) Section 5 demonstrates that the model possesses the same gravitational degrees of freedom and initial value constraints as general relativity, and that subsequent evolution never changes kinetic energies from positive to negative. Like all modified gravity theories, nonlocal cosmology can be differentiated from general relativity with dark energy by how it affects structure formation [33]. Although initial studies revealed no disagreement with the data [34, 35] a more recent analysis by Dodelson and Park [36] has shown that the best current data heavily favors general relativity over the model (1). That crucially important work is reviewed in section 6. In section 7 I argue that models involving a nonlocal invariant other than $\frac{1}{\square}R$ can likely be devised which meet the requirements of structure formation better than general relativity.

2 Causality and Conservation from Nonlocal Actions

In this section we work out how our nonlocal modifications (1) change the Einstein equations. These changes $\Delta G_{\mu\nu}[g](x)$ must be *conserved*, that is their covariant divergence must vanish, $D^\nu \Delta G_{\mu\nu} = 0$. That follows automatically from varying a diffeomorphism invariant action, whether the Lagrangian is local or nonlocal.

The nonlocal corrections we seek must also be *causal*. That is, the variation of $\Delta G_{\mu\nu}[g](x)$ with respect to fields at some point y^μ ,

$$\frac{\delta \Delta G_{\mu\nu}[g](x)}{\delta g_{\rho\sigma}(y)}, \quad (2)$$

must vanish unless y^μ is on or within the past light-cone of x^μ . That cannot follow from the usual effective action by a simple symmetry argument. Consider the self-mass-squared correction to a scalar field $\varphi(x)$ in flat space,

$$\Delta \Gamma[\varphi] = -\frac{1}{2} \int d^4x \int d^4x' \varphi(x) M^2(x; x') \varphi(x'). \quad (3)$$

The scalar self-mass-squared is symmetric under interchange, $M^2(x; x') = M^2(x'; x)$, but even if it vanished for x'^μ outside the past light-cone of x^μ , the effective field equations would still be acausal because the variation affects both the field $\varphi(x)$ in the future and the field $\varphi(x')$ in the past,

$$\frac{\delta \Delta \Gamma[\varphi]}{\delta \varphi(y)} = -\frac{1}{2} \int d^4x \left\{ M^2(y; x) + M^2(x; y) \right\} \varphi(x). \quad (4)$$

One gets causal effective field equations from quantum field theory by employing the Schwinger-Keldysh formalism [37]. Because we are just treating nonlocal models phenomenologically we shall instead enforce causality by resorting to two partial integration tricks. First, consider some functional of the metric times the variation of the nonlocal factor $\frac{1}{\Box} R$ [18],

$$\sqrt{-g} F[g] \frac{\delta}{\delta g^{\mu\nu}} \left(\frac{1}{\Box} R \right) = \sqrt{-g} F[g] \left\{ -\frac{1}{\Box} \frac{\delta \Box}{\delta g^{\mu\nu}} \frac{1}{\Box} R + \frac{1}{\Box} \frac{\delta R}{\delta g^{\mu\nu}} \right\}, \quad (5)$$

$$\longrightarrow -\sqrt{-g} \left(\frac{1}{\Box} F[g] \right) \frac{\delta \Box}{\delta g^{\mu\nu}} \frac{1}{\Box} R + \sqrt{-g} \frac{\delta R}{\delta g^{\mu\nu}} \frac{1}{\Box} F[g]. \quad (6)$$

A similar (illegal) partial integration is needed to vary \Box [18],

$$\begin{aligned} -\sqrt{-g} \left(\frac{1}{\Box} F[g] \right) \frac{\delta \Box}{\delta g^{\mu\nu}} \left(\frac{1}{\Box} R \right) &\longrightarrow \sqrt{-g} \left\{ -\frac{1}{2} g_{\mu\nu} R \left(\frac{1}{\Box} F[g] \right) \right. \\ &\quad \left. -\frac{1}{2} g_{\mu\nu} \left(\frac{1}{\Box} R \right)^{,\rho} \left(\frac{1}{\Box} F[g] \right)_{,\rho} + \left(\frac{1}{\Box} R \right)_{(,\mu} \left(\frac{1}{\Box} F[g] \right)_{,\nu)} \right\}. \end{aligned} \quad (7)$$

Of course it is not correct to go from (5) to (6), nor from left to right in (7), but one does get causal field equations this way provided $\frac{1}{\Box}$ is always defined

with retarded boundary conditions. Conservation is preserved because that depends only upon the relation $\square \times \frac{1}{\square} = 1$, which is valid whether $\frac{1}{\square}$ is retarded or advanced.

We should recall as well the familiar rule for varying the Ricci scalar,

$$\frac{\delta R(y)}{\delta g^{\mu\nu}(x)} = \left[R_{\mu\nu} - D_\mu D_\nu + g_{\mu\nu} \square \right] \delta^4(y-x). \quad (8)$$

With this and our two partial integration tricks (6-7) it is straightforward to work out how $\Delta\mathcal{L}$ changes the field equations. There are three kinds of terms. The first derive from varying the local factors of $R\sqrt{-g}$,

$$\Delta G_{\mu\nu}^a = \left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - D_\mu D_\nu + g_{\mu\nu}\square \right] f\left(\frac{1}{\square}R\right). \quad (9)$$

The second kind come from varying the R inside the function $f(\frac{1}{\square}R)$,

$$\Delta G_{\mu\nu}^b = \left[R_{\mu\nu} - D_\mu D_\nu + g_{\mu\nu}\square \right] \frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right]. \quad (10)$$

And the final kind comes from varying the $\frac{1}{\square}$,

$$\begin{aligned} \Delta G_{\mu\nu}^c = & -\frac{1}{2}g_{\mu\nu}R \frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right] - \frac{1}{2}g_{\mu\nu} \left(\frac{1}{\square}R\right)^{,\rho} \left(\frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right] \right)_{,\rho} \\ & + \left(\frac{1}{\square}R\right)_{(,\mu} \left(\frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right] \right)_{,\nu)}. \end{aligned} \quad (11)$$

Each of the three terms (9), (10) and (11) is manifestly causal if we interpret the factors of $\frac{1}{\square}$ as retarded Green's functions. To see that they are also conserved, note that acting D^ν on each term gives,

$$D^\nu \Delta G_{\mu\nu}^a = -\frac{1}{2}R \partial_\mu f\left(\frac{1}{\square}R\right), \quad (12)$$

$$D^\nu \Delta G_{\mu\nu}^b = \frac{1}{2}R_{,\mu} \frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right], \quad (13)$$

$$D^\nu \Delta G_{\mu\nu}^c = -\frac{1}{2}R_{,\mu} \frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right] + \frac{1}{2} \left(\frac{1}{\square}R\right)_{,\mu} R f'\left(\frac{1}{\square}R\right). \quad (14)$$

The nonlocal addition to the Einstein tensor is,

$$\begin{aligned} \Delta G_{\mu\nu} = & \left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\square - D_\mu D_\nu \right] \left\{ f\left(\frac{1}{\square}R\right) + \frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right] \right\} \\ & + \left[\delta_\mu^{(\rho} \delta_\nu^{\sigma)} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma} \right] \left(\frac{1}{\square}R\right)_{,\rho} \left(\frac{1}{\square} \left[R f'\left(\frac{1}{\square}R\right) \right] \right)_{,\sigma}. \end{aligned} \quad (15)$$

The term $G_{\mu\nu}\{f(\frac{1}{\square}R) + \frac{1}{\square}[Rf'(\frac{1}{\square}R)]\}$ could be viewed as a sort of time-varying Newton constant, with the remaining terms there to enforce conservation.

3 Reconstruction: Λ CDM without Λ

The arbitrary function $f(X)$ in expression (1) is known as the *nonlocal distortion function*. Many models of dark energy and modified gravity have such free functions. The *Reconstruction Problem* consists of adjusting these functions to reproduce a given expansion history, which is usually Λ CDM. The geometry is homogeneous, isotropic and spatially flat,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x} \cdot d\mathbf{x} \quad \Longrightarrow \quad H(t) \equiv \frac{\dot{a}}{a}. \quad (16)$$

For the reconstruction problem we assume $a(t)$ and all its derivatives are known functions of time.

It is straightforward to solve the reconstruction problem for scalar quintessence models [10, 11, 15] and a brief presentation of the solution will help fix ideas. For the geometry (16) the scalar depends just on time and only two of Einstein's equations are nontrivial,

$$3H^2 = 8\pi G \left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi) \right], \quad (17)$$

$$-2\dot{H} - 3H^2 = 8\pi G \left[\frac{1}{2}\dot{\varphi}^2 - V(\varphi) \right]. \quad (18)$$

As usual, G is the Newton constant, and a dot denotes the time derivative. By adding (17) and (18) one obtains the relation,

$$-2\dot{H} = 8\pi G \dot{\varphi}^2. \quad (19)$$

Hence one can reconstruct the scalar's evolution provided the Hubble parameter is monotonically decreasing,

$$\varphi(t) = \varphi_0 \pm \int_0^t dt' \sqrt{\frac{-2\dot{H}(t')}{8\pi G}} \quad \Longleftrightarrow \quad t = t(\varphi). \quad (20)$$

This relation can be inverted (numerically if need be) to give the time as a function of the scalar, $t(\varphi)$. One then determines the potential by subtracting (18) from (17) and evaluating the resulting function of time at $t = t(\varphi)$,

$$V(\varphi) = \frac{1}{8\pi G} \left[\dot{H}(t(\varphi)) + 3H^2(t(\varphi)) \right]. \quad (21)$$

I will review a similar construction for the nonlocal distortion function $f(X)$ that was devised with Cedric Deffayet [38]. It and like techniques [34, 39] allow one to reproduce the Λ CDM expansion history — or any other $a(t)$ without a cosmological constant or dark energy. This is an important distinction between nonlocal cosmology and $f(R)$ models [16].

For the geometry (16) the nontrivial field equations are,

$$3H^2 + \Delta G_{00} = 8\pi G \rho, \quad (22)$$

$$-2\dot{H} - 3H^2 + \frac{1}{3a^2} \delta^{ij} \Delta G_{ij} = 8\pi G p. \quad (23)$$

Here ρ and p are the energy density and pressure, which are assumed to be known functions of time. For example, to reproduce the Λ CDM expansion history without Λ we would use,

$$\rho(t) = \frac{3H_0^2}{8\pi G} \left[\Omega_{\text{mat}} \left(\frac{a_0}{a(t)} \right)^3 + \Omega_{\text{rad}} \left(\frac{a_0}{a(t)} \right)^4 \right], \quad (24)$$

$$p(t) = \frac{3H_0^2}{8\pi G} \left[0 + \frac{1}{3} \Omega_{\text{rad}} \left(\frac{a_0}{a(t)} \right)^4 \right], \quad (25)$$

where a_0 and H_0 are the current values of the scale factor and the Hubble parameter, and Ω_{mat} and Ω_{rad} are the fractions of the current Λ CDM energy density in matter and radiation. The specialization of the nonlocal modification terms to the geometry (16) gives,

$$\begin{aligned} \Delta G_{00} = & \left[3H^2 + 3H\partial_t \right] \left\{ f\left(\frac{1}{\Box}R\right) + \frac{1}{\Box} \left[Rf'\left(\frac{1}{\Box}R\right) \right] \right\} \\ & + \frac{1}{2} \partial_t \left(\frac{1}{\Box}R \right) \times \partial_t \left(\frac{1}{\Box} \left[Rf'\left(\frac{1}{\Box}R\right) \right] \right), \end{aligned} \quad (26)$$

$$\begin{aligned} \Delta G_{ij} = & - \left[2\dot{H} + 3H^2 + 2H\partial_t + \partial_t^2 \right] \left\{ f\left(\frac{1}{\Box}R\right) + \frac{1}{\Box} \left[Rf'\left(\frac{1}{\Box}R\right) \right] \right\} g_{ij} \\ & + \frac{1}{2} \partial_t \left(\frac{1}{\Box}R \right) \times \partial_t \left(\frac{1}{\Box} \left[Rf'\left(\frac{1}{\Box}R\right) \right] \right) g_{ij}. \end{aligned} \quad (27)$$

For this geometry the Ricci scalar is $R(t) = 6\dot{H} + 12H^2$ and the action of $\frac{1}{\Box}$ on some function of time $W(t)$ can be expressed as a double integral from the initial time t_i ,

$$\frac{1}{\Box} [W](t) = - \int_{t_i}^t dt' \frac{1}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') W(t''). \quad (28)$$

The first step of our construction is taking the difference between relations (22) and (23). This leads to a simple equation for the function $F(t)$,

$$F(t) \equiv f\left(X(t)\right) + \frac{1}{\Box} \left[f'\left(X(t)\right) R(t) \right], \quad X(t) \equiv \frac{1}{\Box} [R](t). \quad (29)$$

This differential equation is,

$$\ddot{F} + 5H\dot{F} + (2\dot{H} + 6H^2)F = 8\pi G(\rho - p) - (2\dot{H} + 6H^2). \quad (30)$$

Recognizing that $F = \frac{1}{a^2}$ is a homogeneous solution permits us to reduce the order so that the solution can be expressed in terms of an integral. Changing the dependent variable to $\Phi(t) \equiv a^2(t) \times F(t)$ allows us to express (30) as,

$$\frac{d}{dt} [a^3 \Phi] = a^5 \left[8\pi G(\rho - p) - (2\dot{H} + 6H^2) \right] \equiv a^3(t) \times S(t). \quad (31)$$

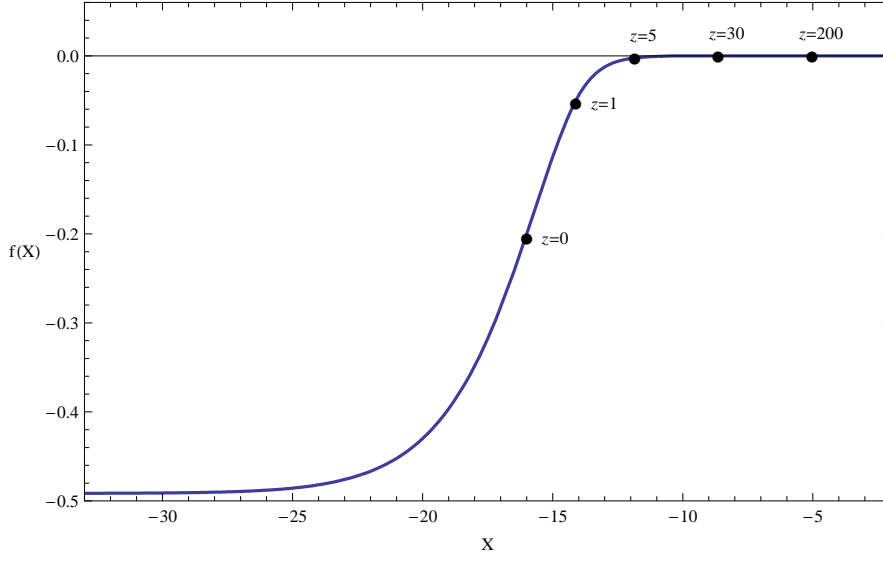


Fig. 1 Plot (solid blue curve) of the reconstructed function $f(X)$ for the nonlocal cosmology reproducing Λ CDM background cosmological evolution, with the same matter content but no cosmological constant. The parameters corresponding to the background cosmology are those of the five-year WMAP release [40]. Circles indicate values of the function $f(X)$ with the corresponding value of the redshift z indicated above.

The solution to (31) for general $a(t)$ can be written as a double integral,

$$\Phi(t) = \Phi_i + \int_{t_i}^t dt' \frac{a_1^3 \dot{\Phi}_1}{a^3(t)} + \int_{t_i}^t dt' \int_{t_i}^{t'} dt'' a^3(t'') S(t''). \quad (32)$$

Requiring no deviation from general relativity at the initial time corresponds to the initial values $\Phi_i = 0 = \dot{\Phi}_i$, which allows us to express the function $F(t)$ using the notation (28),

$$F(t) = -\frac{1}{a^2(t)} \frac{1}{\square} [S](t). \quad (33)$$

The next step is to act $\square \rightarrow -(\partial_t^2 + 3H\partial_t)$ on relation (29) and regard the result as a differential equation for the function $\mathcal{F}(t) \equiv f(X(t))$,

$$\ddot{\mathcal{F}} + (3H - R\dot{X})\dot{\mathcal{F}} = \ddot{F} + 3H\dot{F}. \quad (34)$$

We can find an integrating factor for (34) in terms of the function,

$$Z(t) \equiv \int_{t_i}^t dt' \frac{R(t')}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') R(t''). \quad (35)$$

Multiplying (34) by $a^3 e^Z$ gives a form which can be immediately integrated,

$$\frac{d}{dt} \left[a^3(t) e^{Z(t)} \dot{\mathcal{F}}(t) \right] = e^{Z(t)} \frac{d}{dt} \left[a^3(t) F(t) \right]. \quad (36)$$

The penultimate step is to solve for time as a function of X ,

$$X(t) = - \int_{t_i}^t dt' \frac{1}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') R(t'') \quad \Longleftrightarrow \quad t(X). \quad (37)$$

The desired nonlocal distortion function is $\mathcal{F}(t)$ evaluated at $t = t(X)$,

$$f(X) = \mathcal{F}(t(X)). \quad (38)$$

All of these steps are straightforward to implement numerically for any scale factor $a(t)$. Fig 1 shows the nonlocal distortion function that Deffayet and I [38] constructed for the Λ CDM values pertinent to the Five-Year WMAP data [40]. (Dodelson and Park have carried out the procedure with current data [36].) There is a simple analytic form for $f(X)$ which is indistinguishable from the numerical solution [38],

$$f(X) \approx 0.245 \left[\tanh(0.350Y + 0.032Y^2 + 0.003Y^3) - 1 \right] \quad , \quad Y \equiv X + 16.5. \quad (39)$$

In fact the current data on the expansion history do not require keeping the higher powers of Y in the argument of the tanh function.

4 Perfect Screening for Free

Modified gravity models based on changing R to $f(R)$ suffer from a major problem in that R typically has the same sign for cosmology, where we want big effects to explain the acceleration data, and for the solar system, where significant deviations from general relativity are not permissible. This has prompted the development of elaborate “chameleon mechanisms” in which the extra scalar degree of freedom present in $f(R)$ models is light in cosmological settings and heavy inside the solar system [41]. Nonlocal cosmology differs from $f(R)$ models in two crucial ways:

- There are no extra degrees of freedom to mediate new forces; and
- The factor of \square^{-1} acting on R allows us to define the nonlocal distortion function so that there are no changes from general relativity *at all* in a gravitationally bound system, without affecting cosmology.

The first point will be demonstrated in section 5; it is the second point which concerns us here.

The key fact is that the scalar d’Alembertian $\square \equiv (-g)^{-\frac{1}{2}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$ has different signs when acting on functions of time than on functions of space. This is obvious from the flat space limit $\square \rightarrow -\partial_t^2 + \nabla^2$. In the background cosmology, and small perturbations about it, the time dependence of the Ricci scalar is stronger than its dependence upon space. This means that $\square R$ is typically negative for cosmology. That is clear from the minus sign in expression (37). Indeed, the reconstruction procedure of section 3 determines the nonlocal distortion function $f(X)$ only for $X < 0$. This can be seen from the graph in Fig. 1: reconstruction does not fix the form of $f(X)$ for $X > 0$.

Although gravitationally bound systems are not always static, it is generally true that their dependence upon space is stronger than their dependence upon time. That means $\frac{1}{\Box}R$ is positive inside a gravitationally bound system. Further, reproducing the Λ CDM expansion history requires $f(0) = 0$ [38]. If we wish to completely eliminate any corrections inside gravitationally bound systems it will suffice to define $f(X) = 0$ for all $X > 0$. Hence there is a very simple way to make screening 100% effective inside the solar system, the galaxy, or any other gravitationally bound system, all without affecting the model's behavior for cosmology.

Some model builders consider that requiring $f(X)$ to vanish for $X > 0$ represents a horrendous amount of fine-tuning. This objection would be tenable if one regarded nonlocal cosmology as some sort of fundamental theory, but that is precisely what it cannot be. The initial time t_i we saw in expressions (28), (32), (35) and (37) has no place in a fundamental theory. If the model is to avoid Newton's famous philosophical injunction [20] — and more modern pitfalls [21] — its nonlocal modifications must represent the most cosmologically significant quantum gravitational corrections to the effective field equations from the vast ensemble of infrared gravitons produced during primordial inflation. Those gravitons were of *horizon scale* when they were ripped from the vacuum. (That is why one can study the process using quantum general relativity.) It makes perfect sense for them to have the biggest effect on very large scales, and little or no effect on small scales. This is not exquisite fine-tuning but rather an inevitable and completely natural consequence of the putative physical origin of nonlocal cosmology.

5 Degrees of Freedom and Kinetic Stability

The past two sections have shown how the nonlocal distortion function can be defined to reproduce the Λ CDM expansion history, without changing the predictions of general relativity for gravitationally bound systems. We seek now to understand what sort of initial value formalism the model possesses. In particular, what are its degrees of freedom and constraints? And do any of its degrees of freedom ever become ghosts? These issues were settled by a recent paper written with Stanley Deser [42].

5.1 The Local Version

Shortly after the nonlocal cosmology model (1) was proposed Nojiri and Odinstov devised a local version based on two scalar fields [43]. Their idea is for variation with respect to a scalar ξ to enforce the equation $\Box\phi = R$, so that ϕ becomes $\frac{1}{\Box}R$ plus an arbitrary homogeneous solution. The nonlocal distortion terms in (1) are subject to the modification,

$$Rf\left(\frac{1}{\Box}R\right)\sqrt{-g} \longrightarrow Rf(\phi)\sqrt{-g} + \xi\left(\Box\phi - R\right)\sqrt{-g}. \quad (40)$$

This localized version of the model contains two scalar degrees of freedom which are not present in the original, nonlocal model (1) [44]. The purpose of

this subsection is to demonstrate that one of these extra degrees of freedom is a ghost, as pointed out to me by Gilles Esposito-Farese.

To exhibit the ghost, first partially integrate the scalar kinetic term of (40). Next express this as a difference of two squares,

$$\xi \Box \phi \sqrt{-g} \longrightarrow -\partial_\mu \phi \partial_\nu \xi g^{\mu\nu} \sqrt{-g} , \quad (41)$$

$$= -\frac{1}{4} \partial_\mu (\phi + \xi) \partial_\nu (\phi + \xi) g^{\mu\nu} \sqrt{-g} + \frac{1}{4} \partial_\mu (\phi - \xi) \partial_\nu (\phi - \xi) g^{\mu\nu} \sqrt{-g} . \quad (42)$$

With our spacelike signature one can see that the combination $\phi + \xi$ has positive kinetic energy whereas the combination $\phi - \xi$ is a ghost.

The analysis I have just given is valid so long as the metric is any non-dynamical background. But our metric is of course dynamical — that is the point of modified gravity! — and one might worry about differentiated metric fields from the $Rf(\phi)$ term mixing with the other fields to stabilize the ghost. First, note that such a stabilization would not be phenomenologically acceptable, if it did occur, because $\dot{\phi}(t, \mathbf{x}) - \dot{\xi}(t, \mathbf{x})$ would still develop large inhomogeneities because $\phi - \xi$ is a ghost in *any* background metric. However, the question was analysed in detail by Nojiri, Odintsov, Sasaki and Zhang [45], who concluded that a ghost would form unless,

$$6f'(\phi) > 1 + f(\phi) - \xi > 0 . \quad (43)$$

The authors asserted that classical evolution from special initial conditions could result in these inequalities being preserved for long periods. That is true enough, but it ignores the virulence of kinetic instabilities. Kinetic instabilities are driven by the infinite volume of phase space in the ultraviolet, and they will lead to instantaneous decay if any initial value data permit them. The fact that condition (43) is *linear* in the independent field $\xi(t, \mathbf{x})$ means that the local model cannot be stable.

It might be worried that the existence of this unstable scalar extension of the original model ensures that nonlocal cosmology is itself unstable. Of course the original, nonlocal model is just the local one subject to the initial value constraints $\phi(t_i, \mathbf{x}) = \xi(t_i, \mathbf{x}) = 0 = \dot{\phi}(t_i, \mathbf{x}) = \dot{\xi}(t_i, \mathbf{x})$. Why, it might be wondered, can we not excite the ghost degree of freedom? However, general relativity offers an example in which constraining a particular variable converts an unstable model into a stable one. The variable we have in mind is of course the Newtonian potential, which would be a ghost, and fatal to the theory, were it not a constrained field. So it is at least conceivable that the original, nonlocal model (1) is stable. In the remainder of this section I will demonstrate that it is at indeed free of ghosts.

5.2 Synchronous gauge

Synchronous gauge is the frame of freely falling observers who are released from a spacelike surface with zero relative velocities [46],

$$ds^2 = -dt^2 + h_{ij}(t, \mathbf{x}) dx^i dx^j . \quad (44)$$

Note that $h_{ij}(t, \mathbf{x})$ denotes the full 3-metric, including the cosmological background of $\delta_{ij}a^2(t)$. Synchronous gauge has well-known problems with caustics which should not be important for our purposes. We believe that the basic analysis and conclusions of this section would apply for any lapse and shift.

In synchronous gauge the covariant scalar d'Alembertian takes the form,

$$\square = -\partial_t^2 - \frac{1}{2}h^{ij}\dot{h}_{ij}\partial_t + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}h^{ij}\partial_j\right). \quad (45)$$

Here and henceforth, h^{ij} denotes the inverse of the spatial metric h_{ij} , h stands for the determinant of h_{ij} , and an overdot represents differentiation with respect to time. The various curvatures we require are,

$$R_{00} = -\frac{1}{2}h^{k\ell}\ddot{h}_{k\ell} + \frac{1}{4}h^{ik}h^{j\ell}\dot{h}_{ij}\dot{h}_{k\ell}, \quad (46)$$

$$R_{ij} = \frac{1}{2}\ddot{h}_{ij} + \frac{1}{4}h^{k\ell}\dot{h}_{ij}\dot{h}_{k\ell} - \frac{1}{2}h^{k\ell}\dot{h}_{ik}\dot{h}_{j\ell} + {}^{(3)}R_{ij}, \quad (47)$$

$$R = h^{k\ell}\ddot{h}_{k\ell} + \frac{1}{4}h^{ij}h^{k\ell}\dot{h}_{ij}\dot{h}_{k\ell} - \frac{3}{4}h^{ik}h^{j\ell}\dot{h}_{ij}\dot{h}_{k\ell} + {}^{(3)}R. \quad (48)$$

Note that we follow the usual convention in which a superscript (3) before a quantity denotes its specialization to the purely spatial geometry.

5.3 Initial value data and constraints

Let us first see that the nonlocal field equations require the same initial value data as general relativity, namely, the values of the 3-metric and its first time derivative at $t = t_i$: $h_{ij}(t_i, \mathbf{x})$ and $\dot{h}_{ij}(t_i, \mathbf{x})$. The retarded Green's function associated with \square is defined by the differential equation,

$$\sqrt{h}\square G[h](t, \mathbf{x}; t', \mathbf{x}') = \delta(t-t')\delta^3(\mathbf{x}-\mathbf{x}'), \quad (49)$$

subject to retarded boundary conditions,

$$G[h](0, \mathbf{x}; t', \mathbf{x}') = 0 \quad \forall t' > t. \quad (50)$$

Even though we cannot solve equations (49-50) for an arbitrary 3-metric, their form clearly defines the Green's function $G[h]$ at time t using only the values of h_{ij} and its first time derivative for times less than or equal to t .

Because $\square R$ is the integral of $G[h](t, \mathbf{x}; t', \mathbf{x}')$ multiplied by the Ricci scalar, we need only consider the second time derivatives of the R ; the lower time derivatives and all spatial derivatives are shielded by the inverse differential operator. From expression (48) we see that these second time derivatives can be written in form,

$$R = \partial_t^2 \ln(h) + \frac{1}{4}\left(h^{ij}h^{k\ell} + h^{ik}h^{j\ell}\right)\dot{h}_{ij}\dot{h}_{k\ell} + {}^{(3)}R. \quad (51)$$

Now use relation (45) to express second time derivatives in terms of \square ,

$$\partial_t^2 = -\square - \frac{1}{2}h^{ij}\dot{h}_{ij}\partial_t + \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}h^{ij}\partial_j\right). \quad (52)$$

We can obviously combine relation (52) with (51) to conclude,

$$R = -\square \ln(h) + \frac{1}{4} \left(h^{ik} h^{j\ell} - h^{ij} h^{k\ell} \right) \dot{h}_{ij} \dot{h}_{k\ell} \\ + h^{ij} \left(\Gamma_{ij,k}^k + \Gamma_{ki,j}^k - \Gamma_{k\ell}^k \Gamma_{ij}^\ell - \Gamma_{\ell i}^k \Gamma_{kj}^\ell - \Gamma_{ki}^k \Gamma_{\ell j}^\ell \right). \quad (53)$$

Here $\Gamma_{ij}^k \equiv \frac{1}{2} h^{k\ell} (h_{\ell i,j} + h_{j\ell,i} - h_{ij,\ell})$ is the spatial affine connection and a comma denotes partial differentiation.

With relations (49-50), equation (53) shows that $\frac{1}{\square} R$ involves only the usual initial value data of general relativity: $h_{ij}(0, t_i, \mathbf{x})$ and $\dot{h}_{ij}(t_i, \mathbf{x})$. We can show that these initial value data are apportioned the same way (as general relativity) between constrained fields and gravitational radiation by examining the nonlocal corrections ΔG_{00} and ΔG_{0i} to the constraint equations. Note first from (50) that $\frac{1}{\square}$ and its first time derivative both vanish at $t = t_i$. Further, the nonlocal distortion function vanishes at $X = 0$. So we need only examine the two terms of $\Delta G_{\mu 0}$ in which two covariant derivatives act upon $f(\frac{1}{\square} R) + \frac{1}{\square} [R f'(\frac{1}{\square} R)]$. It is simple to show that neither of the two combinations in the constraint equations contains a second time derivative,

$$g_{00} \square - D_0 D_0 = \frac{1}{2} h^{k\ell} \dot{h}_{k\ell} \partial_t - \frac{1}{\sqrt{h}} \partial_k \left(\sqrt{h} h^{k\ell} \partial_\ell \right), \quad (54)$$

$$g_{0i} \square - D_0 D_i = -\partial_t \partial_i + \frac{1}{2} h^{k\ell} \dot{h}_{ik} \partial_\ell. \quad (55)$$

Hence we conclude that the nonlocal corrections to the constraint equations vanish at $t = t_i$,

$$t = t_i \quad \implies \quad \Delta G_{00} = 0 = \Delta G_{0i}. \quad (56)$$

This means that the nonlocal model possesses the same initial value data as general relativity, and that this initial value data is subject to precisely the same constraints as in general relativity.

5.4 No ghosts

To see that there are no ghosts it suffices to examine the second time derivative terms (in synchronous gauge) of the dynamical equations, $G_{ij} + \Delta G_{ij} = 8\pi G T_{ij}$. The second derivatives of $h_{ij}(t, \mathbf{x})$ in the Einstein tensor follow from expressions (47-48),

$$G_{ij} = \frac{1}{2} \ddot{h}_{ij} - \frac{1}{2} h_{ij} h^{k\ell} \ddot{h}_{k\ell} + O(\partial_t). \quad (57)$$

Of course it is only the first term, $\frac{1}{2} \ddot{h}_{ij}$, which deals with unconstrained fields; the second term represents completely constrained fields. Because general relativity has no ghosts, we need only check that the nonlocal corrections in (15) don't change the sign of the $\frac{1}{2} \ddot{h}_{ij}$ term in (57).

The work of the previous subsection shows that local second time derivatives can only come from the parts of ΔG_{ij} which either multiply G_{ij} or have two covariant derivatives acting upon $f(\frac{1}{\Box}R) + \frac{1}{\Box}[Rf'(\frac{1}{\Box}R)]$. The latter terms are simple to analyze,

$$g_{ij}\Box - D_i D_j = h_{ij}\Box + O(\partial_t) . \quad (58)$$

The local second derivative terms are therefore,

$$G_{ij} + \Delta G_{ij} = \frac{1}{2}\ddot{h}_{ij} \times \left[1 + f\left(\frac{1}{\Box}R\right) + \frac{1}{\Box}\left[Rf'\left(\frac{1}{\Box}R\right)\right] \right] - \frac{1}{2}h_{ij}h^{k\ell}\ddot{h}_{k\ell} \times \left[1 + f\left(\frac{1}{\Box}R\right) + \frac{1}{\Box}\left[Rf'\left(\frac{1}{\Box}R\right)\right] - 4f'\left(\frac{1}{\Box}R\right) \right] + O(\partial_t) . \quad (59)$$

Only the first line of expression (59) represents the unconstrained, dynamical part of h_{ij} . By comparing with the approximate analytic form (39) of the nonlocal distortion function $f(X)$ we see that the coefficient of the dynamical term is reduced at late times, but never by enough to make it change sign. We therefore conclude that no dynamical graviton mode ever becomes a ghost.

6 The Negative Verdict of Structure Formation

Midway through the reconstruction work with Cedric Deffayet [38] it became obvious to me that models of the type (1) were likely to have problems with structure formation because of how they contrive to support late time acceleration without dark energy. To see the problem, specialize the Friedmann equation of general relativity to an energy density comprised of only nonrelativistic matter,

$$3H^2 = 8\pi G\rho \longrightarrow 8\pi G\rho_0\left(\frac{a_0}{a(t)}\right)^3 . \quad (60)$$

Dark energy was posited because the data says the left hand side of (60) is approaching a constant whereas the right hand side is falling off. To understand how nonlocal cosmology resolves this conundrum it suffices to specialize the nonlocal corrections (15) to slowly varying and nearly homogeneous geometries. Under these two assumptions one can drop all the derivatives,

$$\Delta G_{\mu\nu} \approx \left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right] \left\{ f\left(\frac{1}{\Box}R\right) + \frac{1}{\Box}\left[Rf'\left(\frac{1}{\Box}R\right)\right] \right\} . \quad (61)$$

Of course expression (61) is proportional to the Einstein tensor, which means that, for slowly varying and nearly homogeneous geometries, we can regard the modified gravity equations as those of general relativity with a time-varying effective Newton constant,

$$G_{\text{eff}} \equiv \frac{G}{1 + f\left(\frac{1}{\Box}R\right) + \frac{1}{\Box}\left[Rf'\left(\frac{1}{\Box}R\right)\right]} . \quad (62)$$

The right hand side of the modified Friedmann equation ($3H^2 \approx 8\pi G_{\text{eff}}\rho$) is brought into balance with the nearly constant left hand side because the growth in G_{eff} compensates for the redshift of the energy density.

Strengthening the force of gravity is dangerous! The fact that it happens for models of the form (1) convinced me, back in the fall of 2008, that this class of models cannot be correct. However, the problem is not as bad as one might imagine. For example, there is nothing wrong inside the solar system where we have very strong constraints on time variation in Newton's constant. For any gravitationally bound system the geometry is not homogeneous so the approximation (61) is invalid. In any case the effective Newton constant (62) degenerates to G inside any gravitationally bound system because $X \equiv \frac{1}{\Box}R > 0$ and $f(X)$ vanishes for positive X , as we saw in section 4. To encounter a real problem one must study the regime in which perturbations around the cosmological background are small. That is what Dodelson and Park did [35, 36]. I won't attempt to reproduce their work, but I will explain some of the preliminaries and summarize their conclusions.

Dodelson and Park consider scalar perturbations to (16),

$$ds^2 = -\left[1 + 2\Psi(t, \mathbf{x})\right]dt^2 + a^2(t)\left[1 + 2\Phi(t, \mathbf{x})\right]. \quad (63)$$

They treat the potentials $\Psi(t, \mathbf{x})$ and $\Phi(t, \mathbf{x})$ to be small spatial plane waves,

$$\Phi(t, \mathbf{x}) = \tilde{\Phi}(t, \mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \Psi(t, \mathbf{x}) = \tilde{\Psi}(t, \mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (64)$$

They then work out the first order perturbation $\delta(\tilde{G}_\nu^\mu + \Delta\tilde{G}_\nu^\mu) = 8\pi G\delta\tilde{T}_\nu^\mu$ to the nonlocally modified equation for nonrelativistic matter,

$$\delta\tilde{T}_0^0 = -\delta\tilde{\rho}, \quad \delta\tilde{T}_j^i = 0. \quad (65)$$

It helps to see the purely temporal and spatial parts of $\delta\tilde{G}_\nu^\mu$,

$$\delta\tilde{G}_0^0 = -\frac{2k^2}{a^2}\tilde{\Phi} - 6H\dot{\tilde{\Phi}} + 6H^2\tilde{\Phi}, \quad (66)$$

$$\delta\tilde{G}_j^i = \frac{(k^i k^j - k^2 \delta^{ij})}{a^2}[\tilde{\Phi} + \tilde{\Psi}] - 2\delta^{ij}[\ddot{\tilde{\Phi}} + 3H\dot{\tilde{\Phi}} + H\dot{\tilde{\Psi}} - (2\dot{H} + 3H^2)\tilde{\Psi}]. \quad (67)$$

Because expression (67) contains two independent tensor factors, we can infer three equations for the three variables $\tilde{\Phi}(t, \mathbf{k})$, $\tilde{\Psi}(t, \mathbf{k})$ and $\delta\tilde{\rho}(t, \mathbf{k})$. The non-local corrections do not change this overall structure, although they do alter important things such as the general relativistic relation $\tilde{\Phi}(t, \mathbf{k}) = -\tilde{\Psi}(t, \mathbf{k})$ between the two potentials.

I do wish to explain how to perturb functions of $X \equiv \frac{1}{\Box}R$ [48, 35],

$$\delta f(\tilde{X}) = f'(\tilde{X}) \times \delta\tilde{X}, \quad \tilde{X}(t) = -\int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') [6\dot{H}(t'') + 12H^2(t'')], \quad (68)$$

where,

$$\delta\tilde{X}(t, \mathbf{k}) = \int_{t_i}^t dt' G(t, t'; k) \left[\frac{k^2}{a^2} (4\tilde{\Phi} + 2\tilde{\Psi}) + 6\ddot{\tilde{\Phi}} + 6H(4\dot{\tilde{\Phi}} - \dot{\tilde{\Psi}}) + \dot{\tilde{X}}(3\dot{\tilde{\Phi}} - \dot{\tilde{\Psi}}) \right]. \quad (69)$$

The retarded Green's function,

$$G(t, t', k) \equiv -i\theta(t - t')a^3(t') \left[u(t, k)u^*(t', k) - u^*(t, k)u(t', k) \right], \quad (70)$$

is formed from the mode functions of the massless, minimally coupled scalar,

$$\ddot{u} + 3H\dot{u} + \frac{k^2}{a^2}u = 0 \quad \text{and} \quad uu^* - \dot{u}u^* = \frac{i}{a^3}. \quad (71)$$

For the subhorizon modes of interest, Dodelson and Park were able to ignore the various time derivatives in expression (69) and they employed the WKB approximation for the retarded Green's function [35],

$$G(t, t'; k) \approx -\frac{\theta(t - t')a^2(t')}{ka(t)} \sin \left[k \int_{t'}^t \frac{dt''}{a(t'')} \right]. \quad (72)$$

Dodelson and Park found only small changes in the potential $\tilde{\Phi}(t, \mathbf{k})$, however, they discovered that the Newtonian potential $\tilde{\Psi}(t, \mathbf{k})$ grows rapidly from about redshift $z = 1.5$ until it is nearly double that of general relativity at the current time [36]. This is what one expects from a model which strengthens the gravitational force, but it is a phenomenological disaster: weak lensing favors general relativity over nonlocal cosmology by almost 6σ , and redshift space distortions favor general relativity by almost 8σ [36].

7 The Outlook for Nonlocal Modifications of Gravity

Particle theorists are notorious for concocting rosy interpretations of the most devastating phenomenology. At the risk of further damaging our already abysmal reputation, let me say that the recent paper by Dodelson and Park [36] does not depress me, nor does it particularly surprise me. Their excellent work rules out the simplest class of models (1), but I ceased taking those models seriously after realizing that they achieve cosmic acceleration by making the effective Newton constant grow. What one wants is a nonlocal model in which the cosmological constant becomes time dependent, and I eventually succeeded in devising a class of them in collaboration with Nick Tsamis [49]. It seems clear that an elaboration of these models can reproduce the Λ CDM expansion history without disastrously enhancing early structure formation the way the simple models (1) do. In fact the paper by Dodelson and Park encourages me because their larger message is that the latest data slightly favor a universe which is initially more uniform than the one predicted by general relativity [36]. So this is a wonderful opportunity to build modified gravity models which match the data *better* than general relativity, and the Dodelson-Park paper provides crucial information on how to do it.

The class of models which I seek will involve a nonlocal invariant other than $X[g](x) \equiv \frac{1}{\Box} R$. To understand why, I must recount that Tsamis and I have proposed that the bare cosmological constant Λ is not small (so there is no cosmological constant problem) but is today being screened by the self-gravitation between gravitons which were produced during a long epoch of primordial inflation [22]. In the absence of a direct computation from quantum gravity, we attempt to describe the screening process through a general class of nonlocal effective field equations — characterized by another free function $f(-G\Lambda X)$ — which interpolate between the known secular dependence of perturbative corrections [30] to the conjectured regime in which quantum gravitational back-reaction becomes nonperturbatively strong. As long as our function $f(-G\Lambda X)$ grows monotonically and without bound, inflation ends in a characteristic way, and the nonlocal correction terms thereafter settle down to a constant, negative vacuum energy during radiation domination [49].

All of that sounds wonderful, and it is. However, our models respond exactly the wrong way to the onset of matter domination [50]. Instead of engendering a phase of late time acceleration, our nonlocal corrections drive the total vacuum energy negative! This is because the Ricci scalar,

$$R \longrightarrow 6\dot{H} + 12H^2 = 6(2 - \epsilon)H^2, \quad (73)$$

has the same sign for vacuum energy domination ($\epsilon = 0$) and matter domination ($\epsilon = \frac{3}{2}$). So generating an ever-more negative vacuum energy during primordial inflation — which we need to end inflation — means that the same thing also happens after the transition from radiation domination ($\epsilon = 2$) to matter domination.

The cure Tsamis and I proposed [50] is to transform the factor of $-\Lambda$ in the function $f(-G\Lambda X)$, into a curvature invariant which changes sign when the slow roll parameter $\epsilon \equiv -\frac{\dot{H}}{H^2}$ varies from acceleration ($\epsilon < 1$) to deceleration ($\epsilon > 1$). When specialized to the cosmological background (16) a reasonable choice for us is R_{00} ,

$$R_{00} \longrightarrow -3(\dot{H} + H^2) = -3(1 - \epsilon)H^2. \quad (74)$$

The problem is finding an invariant that becomes R_{00} when specialized to the cosmological background (16). The solution Tsamis and I proposed is to construct a timelike 4-velocity field $u^\mu[g](x)$ by normalizing the gradient of $X[g](x) \equiv \frac{1}{\Box} R$ [50],

$$u^\mu[g](x) \equiv \frac{g^{\mu\nu}(x)\partial_\nu X[g](x)}{\sqrt{-g^{\alpha\beta}(x)\partial_\alpha X[g](x)\partial_\beta X[g](x)}}. \quad (75)$$

The desired invariant is obtained by contracting two factors of this into the Ricci tensor. So our new model is based on the replacement [50],

$$f\left(-G\Lambda\frac{1}{\Box}[R]\right) \longrightarrow f\left(G\frac{1}{\Box}[Ru^\mu u^\nu R_{\mu\nu}]\right). \quad (76)$$

There is one more complication to get an acceptable model but that need not concern us here. The main point I want to make is that many features of the extended class of models will carry over from what has already been done for (1), so we do not face the task of beginning anew but rather that of building on the existing foundations.

A final point is that I have also encountered the need for a second invariant in devising relativistic, metric-based extensions of Milgrom's modified gravity Modified Newtonian Dynamics (MOND)[51]. MOND is impressively successful at explaining the structures of galaxies and galactic clusters without dark matter [52], so one suspects that it might represent the Newtonian limit of some fully relativistic, modified gravity model. One can build a non-local model based on $X \equiv \frac{1}{\square} R$ which reproduces the MOND force law[18] — known as the Tully-Fisher relation — but no such model can explain the observed amount of weak lensing [19]. With Cedric Deffayet and Gilles Esposito-Farese I constructed a class of nonlocal models which accomplish both things by involving another invariant [19],

$$Y[g](x) \equiv 4g^{\mu\nu} \left(\partial_\mu \left[u^\alpha u^\beta R_{\alpha\beta} \right] \right) \left(\partial_\nu \left[u^\rho u^\sigma R_{\rho\sigma} \right] \right). \quad (77)$$

One can hardly fail to recognize the analogy,

$$\text{Expansion History} \iff \text{Tully} - \text{Fisher} , \quad (78)$$

$$\text{Structure Formation} \iff \text{Weak Lensing} . \quad (79)$$

Modified gravity models involving only the nonlocal invariant $X[g] \equiv \frac{1}{\square} R$ can meet the requirements of line (78) without dark energy (left) or dark matter (right). However, the requirements of the second line in each case necessitate a second invariant based upon $u^\mu u^\nu R_{\mu\nu}$. Perhaps there is a grand synthesis in which a single nonlocal gravity model describes both cosmology and gravitational forces on large scales, without the need for either dark energy or dark matter?

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